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A Pendant for Pólya: The One-Loop Partition Function of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

Marcus Spradlin and Anastasia Volovich

Kavli Institute for Theoretical Physics
University of California
Santa Barbara, CA 93106 USA
spradlin, nastja@kitp.ucsb.edu

Abstract

We study weakly coupled $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R} \times S^3$ at infinite N, which has interesting thermodynamics, including a Hagedorn transition, even at zero Yang-Mills coupling. We calculate the exact one-loop partition function below the Hagedorn temperature. Our calculation employs the representation of the one-loop dilatation operator as a spin chain Hamiltonian acting on neighboring sites and a generalization of Pólya's counting of 'necklaces' (gauge-invariant operators) to include necklaces with a 'pendant' (an operator which acts on neighboring beads). We find that the one-loop correction to the Hagedorn temperature is $\delta \ln T_{\rm H} = +\lambda/8\pi^2$.

Contents

1. Introduction

The past several years have witnessed a tremendous amount of effort invested into the careful study of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory at large N from a number of complementary approaches. One motivation for much of this work is the fact that this theory is believed to provide, via the AdS/CFT correspondence, the simplest context in which we might hope to understand how to solve large N gauge theories in four dimensions. Optimistically, the apparent integrability [1] of the string theory dual to $\mathcal{N} = 4$ SYM theory

suggests that it might be possible to calculate all physical quantities (at least at infinite N) as exact functions of the 't Hooft parameter $\lambda = g_{\rm YM}^2 N$. The successful calculation of the circular BPS Wilson loop for all λ [2,3] provided an early realization of this hope.

More recently, significant progress has been made on the problem of calculating anomalous dimensions of gauge theory operators. Two related approaches relevant to this problem include the study of semiclassical string solutions following [4], and the study of the dilatation operator directly as the Hamiltonian of an integrable spin chain following [5,6]. Comprehensive reviews of the most recent progress on these approaches can be found in [7,8] and [9]. BMN operators provide a prime example of a class of operators whose anomalous dimensions can be calculated for all λ [10,11]. Very recent work with partial results on the problem of summing up all orders in λ includes [12,13].

In this paper we continue in the fine tradition of chipping away at $\mathcal{N} = 4$ SYM theory from a variety of angles. Our present interest lies in the partition function of the theory on $\mathbb{R} \times S^3$ at infinite N, which displays interesting thermodynamic behavior [14], including a Hagedorn transition [15], even at zero 't Hooft parameter. The partition function can be calculated exactly at $\lambda = 0$ by simply counting gauge-invariant operators using Pólya theory (see for example [15,16,17,18]). Here we calculate the one-loop correction to the partition function below the Hagedorn temperature and find the result

$$\ln \mathrm{Tr}[e^{-\beta H}]\big|_{1-\mathrm{loop}} = -\beta \frac{\lambda}{4\pi^2} \sum_{k=1}^{\infty} \left[k \frac{\langle D_2(x_k) \rangle}{1 - z(x_k)} + \sum_{m=1}^{\infty} \langle P D_2(x_k, x_m) \rangle \right], \qquad (1.1)$$

where $x_n = \omega^{n+1} e^{-\beta n}$, $\omega = e^{2\pi i}$ (so that $\omega^{m/2} = \pm 1$ depending on whether m is even or odd), and the function $z(x)$ is given below in (2.4). The remaining quantities $\langle D_2(x) \rangle$ and $\langle P D_2(w, y) \rangle$ are traces of the one-loop dilatation operator D_2 acting on two neighboring fields inside an operator. We evaluate these traces explicitly in (5.14) and (5.17) below. We also obtain from (1.1) the one-loop correction to the value of the Hagedorn temperature,

$$T_{\rm H}(\lambda) = T_{\rm H}(0) \left(1 + \frac{\lambda}{8\pi^2} + \cdots \right), \qquad T_{\rm H}(0) \approx 0.380 \qquad (1.2)$$

(measured in units where the radius of the S^3 is one), which is presumably another example of an interesting physical quantity which we might hope to one day calculate as an exact function of λ.

We begin in section 2 by reviewing how the free partition function may be calculated by first assembling the partition function for the elementary fields into a partition function

for single-trace operators, and then into the full partition function for multi-trace operators. We discuss the general structure of the one-loop correction to the partition function and comment on the role of operator mixing and $1/N$ effects. In section 3 we rephrase the problem of counting gauge theory operators into the language of spin chains and introduce a generalization of Pólya's necklace problem to what we call necklaces with a 'pendant'. These are necklaces with some local operator $\widehat{O}$, such as a spin chain Hamiltonian, inserted at neighboring beads on the necklace. We derive a general formula expressing a partition function for such necklaces in terms of some basic quantities $\langle O(x)\rangle$ and $\langle PO(w,y)\rangle$ which are easily obtained from $\widehat{O}$. In section 4 we apply this machinery to various familiar subsectors ($SU(2)$, $SO(6)$, and $SL(2)$) of the $\mathcal{N}=4$ SYM theory. This section is mostly a warm-up for section 5, where we calculate $\langle D_2(x)\rangle$ and $\langle PD_2(w,y)\rangle$ for the one-loop dilatation operator D_2 of the full $\mathcal{N}=4$ SYM theory. Finally in section 6 we present our final results for the one-loop corrections to the single- and multi-trace partition functions of $\mathcal{N}=4$ SYM.

2. The $\mathcal{N}=4$ SYM Partition Function

We begin in this section with a discussion of weakly-coupled $\mathcal{N}=4$ SYM theory on $\mathbb{R}\times S^3$ following [15,17]. Since there is more than one way to calculate the free partition function, we review here that derivation which best sets the stage for our calculation of the one-loop correction in the following sections.

2.1. Initial considerations

In general, the partition function is given by

$$\mathcal{Z}(\beta)=\mathrm{Tr}[e^{-\beta H}], \tag{2.1}$$

where β is the inverse temperature and H is the Hamiltonian of the theory on $\mathbb{R}\times S^3$. It is convenient to introduce the bookkeeping parameter

$$x=e^{-\beta}=e^{-1/T}, \tag{2.2}$$

which ranges from $x=0$ (zero temperature) to $x=1$ (infinite temperature). According to the state-operator map there is a one-to-one correspondence between states of the theory on $\mathbb{R}\times S^3$ and gauge-invariant operators on $\mathbb{R}^4$. The Hamiltonian H on $\mathbb{R}\times S^3$ is identified

with the dilatation operator D on the plane, so counting states weighted by x to the power of their energy is equivalent to counting gauge-invariant operators weighted by x to the power of their dimension on the plane. Therefore the partition function can be written as

$$\mathcal{Z}(x) = \mathrm{Tr}[x^D], \tag{2.3}$$

where we implicitly set the radius R of the S^3 to one. The dimensions of quantities such as energy and temperature can be restored by affixing the appropriate power of R.

We start by calculating $\mathcal{Z}(x)$ at tree level, so the dilatation operator reduces to $D = D_0$, which just counts the engineering dimension of an operator. To calculate (2.3) all we need to do is write down a complete basis of operators and count them. The most general gauge-invariant operator can be written as a linear combination of operators with a definite number of traces, and the most general k-trace operator can be expressed as a product of k single-trace operators (keeping in mind that separate traces behave as identical particles and are subject to the appropriate Bose or Fermi statistics). Therefore, a complete basis for arbitrary gauge-invariant operators follows naturally after we specify a complete basis of single-trace operators.

2.2. The $\mathcal{N} = 4$ alphabet

A single-trace operator is a product of the fields $(\phi^I, \lambda^a, F_{\mu\nu})$ of $\mathcal{N} = 4$ SYM theory and their covariant derivatives. Covariant derivatives must always be symmetrized with traces removed, since antisymmetric derivatives can be replaced by field strengths and traces of derivatives give terms which are zero by the equations of motion. Following Polyakov [19] we refer to such objects as the 'letters' of the $\mathcal{N} = 4$ 'alphabet'. We will use $\mathcal{A}$ to denote the collection of letters.

We define $d(A)$ of a letter A to be its engineering dimension, so $d(\phi^I) = 1$, $d(\lambda^a) = \frac{3}{2}$, $d(F_{\mu\nu}) = 2$, and each covariant derivative adds one to the dimension. The enumeration of letters weighted by dimension gives rise to the following elementary partition function for the $\mathcal{N} = 4$ alphabet [15,17],

$$z(x) = \sum_{A \in \mathcal{A}} x^{d(A)} = \frac{2x(3 - \sqrt{x})}{(1 - \sqrt{x})^3}. \tag{2.4}$$

This function has a power series which converges for all temperatures $0 \leq x < 1$ and can be understood as follows:

$$z(x) = \underbrace{6x}_{6(\phi^I)} + \underbrace{16x^{3/2}}_{16(\lambda^a)} + \underbrace{30x^2}_{24(D_\mu \phi^I)+6(F_{\mu\nu})} + \underbrace{48x^{5/2}}_{48(D_\mu \lambda^a)} + \underbrace{70x^3}_{54(D_\mu D_\nu \phi^I)+16(D_\mu F_{\nu\rho})} + \cdots. \tag{2.5}$$

There are only 48 $D_\mu\lambda^a$ instead of 64 because 16 components are set to zero by the Dirac equation. Similarly there are only 54 components of $D_\mu D_\nu \phi^I$ because $D^2\phi^I$ is zero. Finally, 4 of the 24 components of $D_\mu F_{\nu\rho}$ are zero by the equations of motion and another 4 are equal to zero by the Bianchi identity.

In what follows we will frequently need to know various partition functions with the fermion number operator $(-1)^F$ inserted. These factors can be easily dealt with by making use of the fact that bosonic and fermionic operators respectively have integer or half-integer dimensions at tree level. To this end we introduce the quantity

$$\omega = e^{2\pi i}, \tag{2.6}$$

which is equal to $+1$ if it is raised to an integer power and -1 if it is raised to a half-integer power. To see ω in action consider the formula

$$z(\omega x) = \sum_{A\in\mathcal{A}} e^{2\pi i d(A)} x^{d(A)} = \frac{2x(3+\sqrt{x})}{(1+\sqrt{x})^3} = 6x - 16x^{3/2} + \cdots = \sum_{A\in\mathcal{A}} (-1)^{F(A)} x^{d(A)}. \tag{2.7}$$

2.3. Single-trace operators

The next step is to string individual letters of the alphabet $\mathcal{A}$ together to form single-trace operators. The quantity we would like to calculate is the single-trace partition function

$$Z(x) = \sum_{\text{single}-\text{trace } O} x^{D_0(O)}. \tag{2.8}$$

(Note here the notational distinction between the single-trace partition function $Z(x)$ and the full partition function in (2.3), denoted $\mathcal{Z}(x)$.) The only constraint on $O = \mathrm{Tr}[A_1 A_2 \cdots A_L]$ is the overall cyclic invariance of the trace. The enumeration of such single-trace operators is identical to the combinatorial problem of counting the number of distinct necklaces composed of a collection of different types of beads. The solution to this problem, which we review in the next section, may be expressed in terms of $z(x)$ as

$$Z(x) = -z(x) - \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \ln[1 - z(\omega^{n+1} x^n)], \tag{2.9}$$

where $\phi(n)$ is Euler's totient function which counts the number of integers less than n which are relatively prime to n. The first term $-z(x)$ in (2.9) is present simply to subtract away traces of a single letter $\mathrm{Tr}[A]$, since these vanish automatically in the $SU(N)$ theory.

Plugging (2.4) into (2.9) gives an expansion which goes as follows:

$$Z(x) = \underbrace{21x^2}_{21(\mathrm{Tr}[\phi^I\phi^J])} + \underbrace{96x^{5/2}}_{96(\mathrm{Tr}[\phi^I\lambda^a])} + \underbrace{376x^3}_{76(\mathrm{Tr}[\phi^I\phi^J\phi^K])+144(\mathrm{Tr}[\phi^I D_\mu\phi^J])+36(\mathrm{Tr}[\phi^I F_{\mu\nu}])+120(\mathrm{Tr}[\lambda^a\lambda^b])} + \cdots. \tag{2.10}$$

Two comments are in order. The first is that the result (2.9) is only valid in the $N = \infty$ limit of the the $SU(N)$ gauge theory, since we allow arbitrarily high powers of the individual letters. If N were finite, then trace identities would allow a single trace of more than N letters to be reexpressed in terms of higher-trace operators, indicating that the basis of operators we are using would be overcomplete.

The second observation is that the power series expansion of (2.9) converges only for $0 \le x < x_{\mathrm{H}}$, where

$$x_{\mathrm{H}} = 7 - 4\sqrt{3} \approx 0.072. \tag{2.11}$$

The divergence of the partition function at this value, which corresponds to the temperature

$$T_{\mathrm{H}}(\lambda = 0) = -\frac{1}{\ln x_{\mathrm{H}}} = 1/\ln(7 + 4\sqrt{3}) \approx 0.380, \tag{2.12}$$

(measured in units of R^{-1}) has been argued to be the gauge theory dual of the Hagedorn transition in string theory [15,17]. Hagedorn-like behavior in other free large N systems was also observed in [20].

2.4. From single-trace to multi-trace operators

Having determined the partition function $Z(x)$ for single-trace operators, it is an easy combinatoric problem to calculate from this the partition function for an arbitrary number of traces, since the only constraint is that traces should be treated as indistinguishable bosons or fermions. The result is

$$\mathcal{Z}(x) = \exp\left[\sum_{n=1}^{\infty} \frac{Z(\omega^{n+1}x^n)}{n}\right]. \tag{2.13}$$

The partition function $Z_k(x)$ for k-trace operators can be extracted by inserting y^n into the sum and then reading off the coefficient of y^k in the expansion of the exponential.

From the results reviewed above we can see that the complete partition function of $\mathcal{N} = 4$ $SU(N)$ gauge theory at infinite N and zero Yang-Mills coupling is given by the formula

$$\mathcal{Z}(x) = \exp\left[-\sum_{n=1}^{\infty} \frac{z(\omega^{n+1}x^n)}{n}\right] \prod_{n=1}^{\infty} \frac{1}{1 - z(\omega^{n+1}x^n)}. \tag{2.14}$$

The exponential term in (2.14) is (one over) the partition function for gauge group $U(1)$ and the infinite product is the partition function for gauge group $U(N)$, so (2.14) expresses the expected fact that

$$\mathcal{Z}_{SU(N)} = \frac{\mathcal{Z}_{U(N)}}{\mathcal{Z}_{U(1)}}. \tag{2.15}$$

In fact, it is interesting to note that although this analysis has been done at infinite N, the result (2.14) remains correct to all orders in $1/N$ (though certainly not at finite N). This is true because the tree-level dilatation operator D_0 obviously does not receive any $1/N$ corrections, and trace relations are non-perturbative in $1/N$.

2.5. Turning on the Yang-Mills coupling

In (2.14) we have reviewed the complete partition function for $SU(N)$ $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$ at infinite N and zero Yang-Mills coupling. The goal of this paper is to calculate the first-order correction to this result when we turn on the Yang-Mills coupling $g_{\rm YM}$. The dilatation operator D can be expanded in powers of the 't Hooft parameter $\lambda = g_{\rm YM}^2 N$ as

$$D = D_0 + \frac{\lambda}{4\pi^2} D_2 + \cdots, \tag{2.16}$$

so to first order in λ the partition function (2.3) is given by

$$\mathcal{Z}(x, \lambda) = \mathrm{Tr}[x^{D_0 + \frac{\lambda}{4\pi^2} D_2 + \cdots}] = \mathrm{Tr}[x^{D_0}] + \frac{\lambda \ln x}{4\pi^2} \mathrm{Tr}[x^{D_0} D_2] + \cdots. \tag{2.17}$$

Therefore we need to calculate the trace of the one-loop dilatation operator,

$$\mathcal{Z}^{(1)}(x) = \mathrm{Tr}[x^{D_0} D_2]. \tag{2.18}$$

Our calculation will proceed by first calculating the one-loop partition function in the single-trace sector and then assembling together the result for multi-trace operators as in the previous subsection.

2.6. Operator mixing and $1/N$

The tree-level dilatation operator D_0 is diagonal in the trace basis, since the engineering dimension of a k-trace operator is obviously just the sum of the engineering dimensions of the k individual operators. At one loop this is no longer true. Instead we have a formula of the form

$$D_2|k\rangle = \sum_{i=1}^{k} \mathrm{Tr}[O_1]\cdots(D_2\mathrm{Tr}[O_i])\cdots\mathrm{Tr}[O_k] + \frac{1}{N}|k-1\rangle + \frac{1}{N}|k+1\rangle \tag{2.19}$$

for a generic k-trace operator $|k\rangle = \mathrm{Tr}[O_1]\cdots\mathrm{Tr}[O_k]$. At first glance one might be tempted to disregard the $\mathcal{O}(1/N)$ terms since we are working at infinite N. However it is well-known that there are classes of k-trace operators whose one-loop anomalous dimensions receive non-zero contributions from mixing with $k\pm 1$-trace operators [21,22]. This occurs generically for BMN operators, which consist of $L\sim\sqrt{N}$ letters. For such operators the $1/N$ suppression is overwhelmed by the growth of the number of non-planar diagrams [23].

Therefore we will not use $N=\infty$ as a justification to omit the last two terms in (2.19). Fortunately we have an even better excuse, which is simply that these terms are non-diagonal in the trace basis and therefore do not contribute to the quantity (2.18) that we are computing. The situation becomes more complicated starting at two loops, where the correction to the partition function involves $\mathrm{Tr}[x^{D_0}D_4]$ and $\mathrm{Tr}[x^{D_0}D_2^2]$, both of which have some diagonal terms of order $1/N^2$ which can't necessarily be dropped. (In some sense this smells like an 'order of limits' problem, like the one discussed in [24]). Of course, if one focuses on calculating the planar partition function (as opposed to the large N partition function), then none of these issues arise and one can ignore the $1/N$ terms in (2.19) from the beginning. Subtleties in the $1/N$ expansion have been noted in [25].

Note that we have implicitly been using a scalar product on the space of operators which is diagonal in the trace basis,

$$\langle k|l\rangle \propto \delta_{kl}. \tag{2.20}$$

The $\mathcal{N}=4$ SYM theory provides, via the state-operator correspondence, a natural inner product $S_{kl}=\langle k|l\rangle$ on the space of operators which is diagonal in the trace basis at infinite N but receives non-diagonal, trace-mixing corrections beginning at $\mathcal{O}(1/N)$. This operator mixing does not concern us here since the trace

$$\mathrm{Tr}[x^{D_0}D_2] = \sum_{k,l}\langle k|x^{D_0}D_2|l\rangle S^{-1}_{kl} \tag{2.21}$$

is completely independent of the scalar product S. Therefore we are free to choose the most convenient inner product $S_{kl} = \delta_{kl}$. A non-trivial inner product S appeared in studies of $1/N$ corrections to the BMN correspondence, such as [26,27] where matrix elements of the Hamiltonian were compared between gauge theory and string theory. The utility of ignoring the gauge theory inner product for the purpose of calculating basis-independent quantities was emphasized in [28] in the context of calculating eigenvalues of D_2 in the BMN sector.

3. Pólya Necklaces

In this section we develop the machinery which will reduce the calculation of the one-loop partition function

$$Z^{(1)}(x) = \mathrm{Tr}[x^{D_0} D_2] \tag{3.1}$$

in the single-trace sector to the calculation of some elementary traces involving $\widehat{O} = D_2$ acting on two neighboring letters. Roughly speaking, we consider here the problem of 'tracing out' all but two letters of any gauge-invariant operator, and express the result in terms of traces of $\widehat{O}$ over the remaining two letters. This is accomplished by translating the calculation into the language of spin chains, and then using a generalization of Pólya's counting theory.

3.1. Free necklaces

A necklace N of length L is a collection of L objects $(A_1 \cdots A_L)$ chosen from some fixed set $A \in \mathcal{A}$ of 'beads', such that two necklaces are identified if they differ from each other by a cyclic rotation. It is useful to introduce a counting function d on the beads, and define d to act additively on the beads of a necklace,

$$d(A_1 \cdots A_L) = \sum_{i=1}^{L} d(A_i). \tag{3.2}$$

The analysis of this section will be general, but of course for the desired application to $\mathcal{N} = 4$ SYM theory we remember in the back of our mind that we will take $\mathcal{A}$ to be the $\mathcal{N} = 4$ alphabet, and $d(A)$ will be the engineering dimension of the letter A.

A central result of Pólya's counting theory is that the necklace partition function

$$Z(x) = \sum_{N} x^{d(N)}, \tag{3.3}$$

where we sum over necklaces N of arbitrary length $L \geq 1$, is given by

$$Z(x) = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \ln[1 - z(x^n)], \tag{3.4}$$

where $z(x)$ is the generating function for the beads,

$$z(x) = \sum_{A \in \mathcal{A}} x^{d(A)}. \tag{3.5}$$

The formula (3.4) is valid when the beads are all bosons. The generalization to include fermions is straightforward and will be presented below.

3.2. Necklaces with a pendant

Instead of reviewing the elementary derivation of the result (3.4), we will consider a useful generalization from which we will recover (3.4) as a special case. To describe the generalization that we are interested in, it is useful to think of a necklace of length L as a spin chain of length L, where on each site of the spin chain the spin vector takes values in the set $\mathcal{A}$. Of course we have the constraint that only cyclically invariant spin configurations correspond to necklaces. Therefore, we can recast the calculation of (3.3) into spin chain language by writing the partition function as

$$Z(x) = \sum_{N} x^{d(N)} = \sum_{L=1}^{\infty} \mathrm{Tr}_L[\mathcal{P} x^d], \tag{3.6}$$

where Tr_L denotes the trace over spin chains of length L and $\mathcal{P}$ denotes the projection operator onto the subspace of cyclically invariant spin configurations. The projector can be written explicitly as

$$\mathcal{P} = \frac{1}{L}(1 + T + T^2 + \cdots + T^{L-1}), \tag{3.7}$$

where T is the translation operator which sends site i on the chain to site $i+1$ and satisfies $T^L = 1$.

The generalization of $\mathrm{Tr}_L[\mathcal{P} x^d]$ that we would like to consider is to the case

$$\mathrm{Tr}_L[\mathcal{P} x^d \widehat{O}], \tag{3.8}$$

where $\widehat{O}$ is any homogeneous operator which commutes with d and which acts only on two neighboring sites at a time, so that it may be decomposed into the form

$$\widehat{O} = \sum_{i=1}^{L} \widehat{O}_{i,i+1}, \qquad \widehat{O}_{i,i+1} = 1_1 \otimes \cdots \otimes 1_{i-1} \otimes O \otimes 1_{i+2} \otimes \cdots \otimes 1_L. \tag{3.9}$$

Spin chain Hamiltonians are of course prime examples of such operators, and eventually we will apply the present machinery to the case $\widehat{O} = D_2$, but our analysis will continue to be as general as possible. Since the operator O only connects two neighboring sites at a time, but can slide all the way around the necklace, we can think of O as a 'pendant' hanging from two adjacent beads on the necklace.

Given any such operator O, an obvious quantity of interest is the expectation value

$$\langle O(x)\rangle = \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[x^d O] = \sum_{A_1,A_2\in\mathcal{A}} x^{d(A_1)+d(A_2)}\langle A_1A_2|O|A_1A_2\rangle. \tag{3.10}$$

We will see that knowing $\langle O(x)\rangle$ is almost, but not quite enough information to allow for the calculation of (3.8). The other quantity that we will need to know is the 'permuted trace'

$$\langle PO(w,y)\rangle = \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[Pw^{d_1}y^{d_2}O] = \sum_{A_1,A_2\in\mathcal{A}} w^{d(A_1)}y^{d(A_2)}\langle A_1A_2|O|A_2A_1\rangle, \tag{3.11}$$

where P is the permutation operator on $\mathcal{A}\times\mathcal{A}$ and the two sites are counted with different variables w and y. Let us now see how to reduce the calculation of (3.8) to the calculation of these two quantities.

Since $\mathcal{P}$ projects onto cyclically invariant states anyway, the sum over i in (3.9) is actually redundant. We can affix the pendant to sites $(i, i+1) = (1, 2)$, and the sum in (3.9) just gives a factor of L which cancels the $1/L$ in (3.7) giving

$$\mathrm{Tr}_L[\mathcal{P}x^d\widehat{O}] = \sum_{k=0}^{L-1}\mathrm{Tr}_L[T^k x^d\widehat{O}_{12}]. \tag{3.12}$$

This trace may be expressed as

$$\begin{aligned}\mathrm{Tr}_L[\mathcal{P}x^d\widehat{O}] &= \sum_{k=0}^{L-1}\sum_{A_1,\ldots,A_L}\langle A_1\cdots A_L|T^k x^d\widehat{O}_{12}|A_1\cdots A_L\rangle \\ &= \sum_{k=0}^{L-1}\sum_{A_1,\ldots,A_L} x^{d(A_1)+\cdots+d(A_L)}\langle A_1A_2|O|A_{1+k}A_{2+k}\rangle\prod_{i=3}^{L}\delta_{A_i,A_{i+k}}.\end{aligned} \tag{3.13}$$

Upon contemplating the formula (3.13), it is clear that after we sum over $A_3,\ldots,A_L$, only two possible structures can emerge,

$$(i)\ \ \langle A_1A_2|O|A_1A_2\rangle, \qquad \text{or} \qquad (ii)\ \ \langle A_1A_2|O|A_2A_1\rangle, \tag{3.14}$$

depending on whether k and L are such that the Kronecker delta functions in (3.13) end up connecting site $1+k$ to site 1 or to site 2. For example, $k=0$ clearly gives the former structure while $k=1$ clearly gives the latter. But for general k and L, what is the criterion which tells us which of the two possibilities (3.14) we get?

Let us start with site $1+k$ and follow it around the necklace using n of the Kronecker delta functions,

$$1+k \to 1+2k \to 1+3k \to \cdots \to 1+(n+1)k. \tag{3.15}$$

Since there are only $L-2$ delta functions in total, we have $0 \le n \le L-2$. Now let $m=(L,k)$ be the greatest common divisor of L and k. If $m>1$, then by step $n=L/m-1$ at the latest, site $1+k$ will have been connected to site

$$1+k \to 1+(n+1)k = 1+(L/m-1+1)k = 1+(k/m)L = 1 \bmod L. \tag{3.16}$$

On the other hand, if k and L are relatively prime ($m=1$), then there is no $n<L$ such that $1+(n+1)k=1 \bmod L$, so $1+k$ cannot get connected to site 1 and hence must be connected to site 2. We conclude that we get the trace structure of type (ii) if and only if k and L are relatively prime, and we get structure (i) otherwise. Let us therefore study these cases separately.

3.3. Case (i): k and L have a common factor

As a warm-up exercise let us consider the case $L=15$, $k=6$, so that $m=(15,6)=3$. Then as we sum over sites $3,4,\cdots,L$, the delta functions in (3.13) sew together the beads of the necklace as follows:

$$\begin{aligned} &7 \to 13 \to 4 \to 10 \to 1, \\ &8 \to 14 \to 5 \to 11 \to 2, \\ &9 \to 15 \to 6 \to 12 \to 3 \to 9. \end{aligned} \tag{3.17}$$

Each line in this formula can be though of as a 'strand' of the necklace. So the necklace with $L=15$ and $k=6$ is composed of $m=(15,6)=3$ strands, and the Kronecker delta functions in (3.13) force all of the beads on any given strand to be the same.

The strands starting with $1+k$ and $2+k$ end up respectively at sites 1 and 2, confirming our earlier analysis. The third line in (3.17) denotes the following contribution to (3.13):

$$\begin{aligned} &\sum_{A_3,A_9,A_{15},A_6,A_{12}} x^{d(A_3)+d(A_9)+d(A_{15})+d(A_6)+d(A_{12})} \delta_{A_3,A_9}\delta_{A_9,A_{15}}\delta_{A_{15},A_6}\delta_{A_6,A_{12}}\delta_{A_{12},A_3} \\ &\quad = \sum_{A_3} x^{5d(A_3)} = z(x^5). \end{aligned} \tag{3.18}$$

The first two strands in (3.17) involve the sites 1 and 2 where the pendant is attached, and it is not hard to see that they end up contributing a factor of

$$\sum_{A_1,A_2} x^{5d(A_1)+5d(A_2)} \langle A_1 A_2 | O | A_1 A_2 \rangle = \langle O(x^5) \rangle, \tag{3.19}$$

using the definition (3.10). Combining these results, we find that the total contribution to the sum (3.13) for the case $L = 15$ and $k = 6$ is

$$z(x^5)\, \langle O(x^5) \rangle. \tag{3.20}$$

The generalization of this analysis is straightforward. A necklace with general L and k will have $m = (L, k)$ different strands, each with L/m beads. Two of those strands (like the first two in (3.17)) will involve the sites 1 and 2 and give rise to

$$\langle O(x^{L/m}) \rangle. \tag{3.21}$$

The remaining $m - 2$ strands (note that we are assuming here that $m \geq 2$), like the last line in (3.17), give a factor of

$$(z(x^{L/m}))^{m-2}. \tag{3.22}$$

Combining (3.21) and (3.22), we conclude that the total contribution to (3.13) from all k such that $m = (k, L) > 1$ is

$$\sum_{\substack{k=0 \\ m=(k,L)>1}}^{L-1} \left[z(x^{L/m}) \right]^{m-2} \langle O(x^{L/m}) \rangle. \tag{3.23}$$

3.4. Case (ii): k and L are relatively prime

As a warm-up exercise let us consider the case $L = 14$, $k = 5$. The Kronecker delta functions in (3.13) now sew together the beads

$$\begin{aligned} &6 \to 11 \to 2, \\ &7 \to 12 \to 3 \to 8 \to 13 \to 4 \to 9 \to 14 \to 5 \to 10 \to 1, \end{aligned} \tag{3.24}$$

confirming our general analysis that site $1 + k$ gets connected to 2, and site $2 + k$ gets connected to 1, giving trace structure (ii). The first line indicates a contribution of

$$\sum_{A_6,A_{11}} \delta_{A_6,A_{11}} \delta_{A_{11},A_2} x^{d(A_6)+d(A_{11})+d(A_2)} = x^{3d(A_2)}, \tag{3.25}$$

while the second line similarly denotes a contribution of $x^{11d(A_1)}$. Putting everything together, we find that for $L = 14$ and $k = 5$, (3.13) reduces to

$$\sum_{A_1,A_2} x^{11d(A_1)+3d(A_2)} \langle A_1 A_2 | O | A_2 A_1 \rangle = \langle PO(x^{11}, x^3) \rangle, \tag{3.26}$$

using the definition (3.11).

The generalization to arbitrary k and L is straightforward. After $n-1$ steps in the first line of (3.24), site $1+k$ will connect to site $1+nk$. Therefore, the length of the first strand is the smallest non-negative n such that $1 + nk = 2 \mod L$, or equivalently $nk = 1 \mod L$. The total contribution to (3.13) from all k which are relatively prime to L is therefore

$$\sum_{\substack{k=0 \\ (k,L)=1}}^{L-1} \langle PO(x^{L-n(L,k)}, x^{n(L,k)}) \rangle, \qquad n(L,k) = \min\{n \geq 0 : nk = 1 \mod L\}. \tag{3.27}$$

Solving $nk = 1 \mod L$ for fixed k and L is equivalent to finding n, m such that $nk - Lm = 1$. Given that k is relatively prime to L, it is clear that a solution exists only if n and L are also relatively prime. Moreover, it is clear that $n \leq L$ (otherwise subtracting L from n would give a smaller solution). Finally, the set of $\{k : (k, L) = 1\}$ is in one-to-one correspondence with the set of $n(L,k)$, simply because the condition $nk = 1 \mod L$ is symmetric in n and k. Therefore, although $n(L,k)$ is not generically equal to k, the sum in (3.27) is equivalent to[1]

$$\sum_{\substack{k=0 \\ (k,L)=1}}^{L-1} \langle PO(x^{L-k}, x^k) \rangle. \tag{3.28}$$

3.5. Summary and main result

We now combine the contributions (3.23) and (3.28), writing the result as

$$\begin{aligned} \mathrm{Tr}_L[\mathcal{P} x^d \widehat{O}] = \sum_{k=0}^{L-1} \left[z(x^{L/(k,L)}) \right]^{(k,L)-2} \langle O(x^{L/(k,L)}) \rangle \\ + \sum_{\substack{k=0 \\ (k,L)=1}}^{L-1} \left[\langle PO(x^{L-k}, x^k) \rangle - z(x^L)^{-1} \langle O(x^L) \rangle \right], \end{aligned} \tag{3.29}$$

[1] We are grateful to Jan Plefka for pointing out a flaw in our original proof of this formula.

where in the first term we omitted the constraint $m > 1$ from (3.23) at the expense of subtracting off the extra terms on the second line. Now we can trade the sum over k in the first term of (3.29) for a sum over divisors a of L, to write

$$\mathrm{Tr}_L[\mathcal{P}x^d\widehat{O}] = \sum_{a|L}\phi(a)[z(x^a)]^{L/a-2}\langle O(x^a)\rangle + \sum_{\substack{k=0\\(k,L)=1}}^{L-1}\left[\langle PO(x^{L-k},x^k)\rangle - z(x^L)^{-1}\langle O(x^L)\rangle\right]. \tag{3.30}$$

At this step let us pause for a moment to explain how to recover the Pólya formula (3.4) as promised. To this end we need to consider the special case where the operator O is proportional to the identity matrix, and specifically $O = 1/L$. This is the correct normalization which gives rise to $\widehat{O} = 1$ when plugged into the sum over sites in (3.9). For $O = 1/L$ we easily find

$$\langle O(x)\rangle = \frac{1}{L}z(x)^2, \qquad \langle PO(w,y)\rangle = \frac{1}{L}z(wy). \tag{3.31}$$

The second term in (3.30) therefore drops out, leaving just

$$\mathrm{Tr}_L[\mathcal{P}x^d] = \frac{1}{L}\sum_{a|L}\phi(a)[z(x^a)]^{L/a}. \tag{3.32}$$

The sum over L is performed in the usual way, and we obtain

$$\sum_{L=1}^{\infty}\mathrm{Tr}_L[\mathcal{P}x^d] = -\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\ln[1-z(x^n)], \tag{3.33}$$

which is the desired result (3.4).

Having confirmed that the formula (3.30) reduces to the known answer for the special case $\widehat{O} = 1$, let us now consider operators O which do not depend explicitly on L. In particular this implies, through (3.9), that the eigenvalues of $\widehat{O}$ scale linearly with L. Then we can perform the sum over $L > 1$ to arrive at

$$\begin{aligned}\sum_{L=2}^{\infty}\mathrm{Tr}_L[\mathcal{P}x^d\widehat{O}] = -\frac{\langle O(x)\rangle}{z(x)} + \sum_{n=1}^{\infty}\phi(n)\frac{\langle O(x^n)\rangle}{z(x^n)}\frac{1}{1-z(x^n)} \\ + \sum_{L=2}^{\infty}\sum_{\substack{k=0\\(k,L)=1}}^{L-1}\left[\langle PO(x^{L-k},x^k)\rangle - \frac{\langle O(x^L)\rangle}{z(x^L)}\right].\end{aligned} \tag{3.34}$$

The first term is present to subtract off the part of the second term which would correspond to $L = 1$, which we omit since it can't support a pendant (and moreover is irrelevant in

the $SU(N)$ gauge theory). A final step is to simplify (3.34) by changing the summation variable from n to L and combining everything into the main result

$$\sum_{L=2}^{\infty} \mathrm{Tr}_L[\mathcal{P}x^d\widehat{O}] = \sum_{L=1}^{\infty} \sum_{k:(k,L)=1} \left[\frac{\langle O(\omega^{L+1}x^L)\rangle}{1-z(\omega^{L+1}x^L)} + \delta_{L\neq 1}\langle PO(\omega^{L-k+1}x^{L-k}, \omega^{k+1}x^k\rangle \right]. \tag{3.35}$$

It is a straightforward exercise to generalize the analysis of the previous subsections to allow for fermionic beads, and we have included here the appropriate factors of ω which keep track of the minus signs appearing when such beads are permuted. (The permutation operator P is understood to be graded, i.e. $P|A_1A_2\rangle = (-1)^{F_1F_2}|A_2A_1\rangle$.)

The first term in (3.35) for $L = 1$ is precisely what one would obtain by making the crude estimate that the only effect of the projection $\mathcal{P}$ onto cyclically invariant states is to insert a factor of $1/L$. The rest of (3.35) is the detailed correction to this approximation.

In all the cases relevant to $\mathcal{N} = 4$ SYM theory that we study below, we will see that the second term in (3.35) is a very small correction in the sense that its contribution to the coefficient of x^n is negligible for large n. In particular, we will find that $\langle O(x)\rangle$ and $\langle PO(w, y)\rangle$ converge for all temperatures so that the large temperature behavior of (3.35) is dominated by the pole $1/(1 - z(x))$ in the first term.

4. Examples

We can gain some insight into the formula (3.35) by applying it to some subsectors of the gauge theory. The implication of the results presented in this section for the thermodynamics of $\mathcal{N} = 4$ SYM theory is unclear since there is no sense in which the sectors decouple from each other at finite temperature (we do not consider here the addition of chemical potentials for various charges). However, we believe this section is a useful warm-up exercise for the more complicated analysis which follows. Moreover, the results given here for the traces of the $SU(2)$, $SO(6)$ and $SL(2)$ spin chain Hamiltonians may be of interest from the point of view of integrability in those sectors. An additional subsector which is of independent interest is the $SU(2|4)$ subsector, which at one loop is isomorphic to the 't Hooft large N limit of the plane-wave matrix model [10]. This subsector is considered in [29].

4.1. The $SU(2)$ subsector

This subsector consists of all operators of the form

$$\mathrm{Tr}[XXZZZXZZ\cdots XXZ], \tag{4.1}$$

where X and Z are two holomorphic scalar fields. The 'alphabet' for this sector is $\mathcal{A} = \{X, Z\}$, the dimension formula is $d(A) = 1$, and the elementary partition function is $z(x) = 2x$. In the natural basis $\{XX, XZ, ZX, ZZ\}$ for $\mathcal{A} \times \mathcal{A}$, the matrix elements of the permutation operator P and the one-loop Hamiltonian D_2 are simply

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_2 = \frac{1}{2}(1 - P). \tag{4.2}$$

We immediately find

$$\langle D_2(x) \rangle = x^2, \qquad \langle P D_2(w, y) \rangle = -wy. \tag{4.3}$$

Plugging these into the main formula (3.35) gives, after some simplification, the following formula for the trace of the Hamiltonian in the $SU(2)$ sector:

$$\mathrm{Tr}[x^{D_0} D_2] = x - \sum_{n=1}^{\infty} \phi(n) x^n \left(\frac{1 - 3x^n}{1 - 2x^n} \right). \tag{4.4}$$

As a check, we used a computer to calculate the trace of the $SU(2)$ spin chain Hamiltonian for all chains of length $L \leq 26$ and successfully matched the coefficients in the expansion of (4.4) up to order x^{26}.

4.2. The $SO(6)$ subsector

This subsector consists of all operators built only out of scalar fields with no derivatives,

$$\mathrm{Tr}[\phi^{I_1} \cdots \phi^{I_L}], \qquad I_i = \{1, \ldots, 6\}. \tag{4.5}$$

The alphabet is $\mathcal{A} = \{\phi^1, \ldots, \phi^6\}$, the dimension formula is $d(A) = 1$, and the elementary partition function is $z(x) = 6x$. In the natural basis $|I_1 I_2\rangle = \phi^{I_1} \phi^{I_2}$ for $\mathcal{A} \times \mathcal{A}$, the matrix elements of D_2 and P are [5]

$$\begin{aligned} \langle I_1 I_2 | D_2 | J_1 J_2 \rangle &= \frac{1}{4} \delta_{I_1 I_2} \delta_{J_1 J_2} + \frac{1}{2} \delta_{I_1 J_1} \delta_{I_2 J_2} - \frac{1}{2} \delta_{I_1 J_2} \delta_{I_2 J_1}, \\ \langle I_1 I_2 | P | J_1 J_2 \rangle &= \delta_{I_1 J_2} \delta_{I_2 J_1}. \end{aligned} \tag{4.6}$$

A simple calculation yields

$$\langle D_2(x)\rangle = \frac{33}{2}x^2, \qquad \langle PD_2(w,y)\rangle = -\frac{27}{2}wy, \tag{4.7}$$

which leads to the result

$$\mathrm{Tr}[x^{D_0}D_2] = \frac{27}{2}x - \frac{3}{2}\sum_{n=1}^{\infty}\phi(n)x^n\left(\frac{9-65x^n}{1-6x^n}\right) \tag{4.8}$$

for the trace of the $SO(6)$ Hamiltonian. As a check, we used a computer to calculate the trace of the $SO(6)$ spin chain Hamiltonian for all chains of length $L \leq 11$ and successfully matched the coefficients in the expansion of (4.8) up to order x^{11}.

4.3. The $SL(2)$ subsector

This subsector consists of all operators of the form

$$\mathrm{Tr}[D^{i_1}Z\cdots D^{i_L}Z], \qquad i_n \in \{0,1,\ldots\}, \tag{4.9}$$

where Z is a single holomorphic scalar field and D is a single holomorphic covariant derivative. The alphabet is $\mathcal{A} = \{Z, DZ, D^2Z, \cdots\}$, the dimension formula is $d(D^iZ) = i+1$, and the elementary partition function is

$$z(x) = \sum_{A\in\mathcal{A}} x^{d(A)} = \sum_{i=0}^{\infty} x^{i+1} = \frac{x}{1-x}. \tag{4.10}$$

In the basis $|i_1 i_2\rangle = D^{i_1}Z\,D^{i_2}Z$ for $\mathcal{A}\times\mathcal{A}$ the matrix elements of D_2 are [30]

$$\langle i_1 i_2|D_2|j_1 j_2\rangle = \frac{1}{2}\delta_{i_1+i_2,j_1+j_2}\left[\delta_{i_1,j_1}(h(j_1)+h(j_2)) - \frac{\delta_{i_1\neq j_1}}{|i_1-j_1|}\right], \tag{4.11}$$

where $h(j)$ are the harmonic numbers

$$h(j) = \sum_{n=1}^{j}\frac{1}{n}, \qquad h(0) = 0. \tag{4.12}$$

The matrix elements of P are obviously

$$\langle i_1 i_2|P|j_1 j_2\rangle = \delta_{i_1 j_2}\delta_{i_2 j_1}. \tag{4.13}$$

A simple calculation yields

$$\langle D_2(x)\rangle = -\frac{x^2}{(1-x)^2}\ln(1-x) \tag{4.14}$$

and

$$\langle PD_2(w,y)\rangle = \frac{1}{2}\frac{wy}{1-wy}\ln\left[\frac{(1-w)(1-y)}{(1-wy)^2}\right]. \tag{4.15}$$

After some simplification, we find for the trace of the $SL(2)$ Hamiltonian the result

$$\mathrm{Tr}[x^{D_0}D_2] = -\sum_{n=1}^{\infty}\phi(n)\frac{x^n}{1-2x^n}\ln(1-x^n) + \sum_{L=1}^{\infty}\frac{x^L}{1-x^L}\sum_{\substack{k=1\\(k,L)=1}}^{L}\ln(1-x^k). \tag{4.16}$$

As a check, we used a computer to calculate the trace of the $SL(2)$ spin chain Hamiltonian for all chains with total dimension $D_0 \leq 19$ and successfully matched the coefficients in the expansion of (4.16) up to order x^{19}.

5. Traces of the $PSL(4|4)$ Spin Chain Hamiltonian

We now turn to our next step, which is to apply the result (3.35) to the calculation of the one-loop correction to the partition function of $\mathcal{N}=4$ SYM theory on $\mathbb{R}\times S^3$ in the single-trace sector:

$$Z^{(1)}(x) = \mathrm{Tr}[x^{D_0}D_2]. \tag{5.1}$$

To this end, we compute in this section the quantities

$$\langle D_2(x)\rangle = \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[x^{D_0}D_2], \qquad \langle PD_2(w,y)\rangle = \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[Pw^{D_{0(1)}}y^{D_{0(2)}}D_2] \tag{5.2}$$

needed to invoke (3.35) for the full $PSL(4|4)$ spin chain Hamiltonian D_2.

The calculation of $\langle D_2(x)\rangle$ is greatly facilitated by making use of the $PSL(4|4)$ symmetry of $\mathcal{N}=4$ SYM theory. Since the dilatation operator D_2 commutes with this symmetry, the action of D_2 on an arbitrary state can be decomposed into its action on irreducible representations of $PSL(4|4)$. We therefore begin with a discussion of the relevant representations. Unfortunately, the operator $w^{D_{0(1)}}y^{D_{0(2)}}$ does not commute with the two-letter $PSL(4|4)$ Casimir, so the calculation of $\langle PD_2(w,y)\rangle$ will prove more difficult.

5.1. Digraphs in the $\mathcal{N}=4$ language

The elementary fields and their covariant derivatives which make up the alphabet $\mathcal{A}$ of $\mathcal{N}=4$ SYM theory constitute the so-called 'singleton' representation of $PSL(4|4)$. The superconformal primary state is the scalar field ϕ^I, with quantum numbers

$$[0,1,0]_{(0,0)} \tag{5.3}$$

under $SL(4)$ flavor rotations and the $SL(2)\times SL(2)$ Lorentz algebra. The singleton representation is frequently denoted $\mathcal{V}_{\rm F}$, although we shall continue to refer to it as $\mathcal{A}$ for consistency.

Since the one-loop dilatation operator D_2 only acts on two letters at a time and commutes with the two-letter Casimir $J^2_{(12)}$ of $PSL(4|4)$, it is sufficient to consider the decomposition of the product of two copies of the singleton representation into irreducible representations of $PSL(4|4)$. The decomposition is

$$\mathcal{A}\times\mathcal{A}=\sum_{j=0}^{\infty}\mathcal{V}_j, \tag{5.4}$$

where $\mathcal{V}_j$ is the module whose superconformal primary is an eigenstate of the $PSL(4|4)$ Casimir with eigenvalue $j(j+1)$ and quantum numbers

$$[0,2,0]_{(0,0)}, \qquad [1,0,1]_{(0,0)}, \qquad \text{and} \qquad [0,0,0]_{(\frac{j}{2}-1,\frac{j}{2}-1)} \text{ for } j\geq 2. \tag{5.5}$$

In the notation of [31], we have

$$\begin{aligned}
\mathcal{A} &= \mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,1,0](0,0)},\\
\mathcal{V}_0 &= \mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,2,0](0,0)},\\
\mathcal{V}_1 &= \mathcal{B}^{\frac{1}{4},\frac{1}{4}}_{[1,0,1](0,0)},\\
\text{and } \mathcal{V}_j &= \mathcal{C}^{1,1}_{[0,0,0](\frac{1}{2}j-1,\frac{1}{2}j-1)} \text{ for } j\geq 2.
\end{aligned} \tag{5.6}$$

In linguistics, a group of two successive letters whose phonetic value is a single sound, such as *ng* in *Yang* or *th* in *theory*, is called a 'digraph', so we can think of the $\mathcal{V}_j$ as the digraphs of $\mathcal{N}=4$ Yang-Mills theory.

It is straightforward to count the primary states in $\mathcal{V}_j$, weighted in the usual way by x^{D_0}. For $j=0$ and $j\geq 2$ the results can be read off respectively from (6.13) and (5.45) of [31], or from tables 7 and 8 of [18]. We did not immediately find the primary content of $\mathcal{V}_1$

in the literature, but the derivation thereof is straightforward and the result is presented in appendix A. The results for all $\mathcal{V}_j$ can be summarized in the expressions

$$\begin{aligned} V_0(x) &= 4x^2(1+\sqrt{x})^4(5-x), \\ V_j(x) &= x^j(1+\sqrt{x})^7(j-1+(j+2)\sqrt{x})(j-1+5\sqrt{x}-(j+2)x), \qquad j \geq 1. \end{aligned} \tag{5.7}$$

Setting $x = 1$ counts the total number of primaries in $\mathcal{V}_j$, which is $2^8(2j+1)$ for any $j \geq 0$.

Note that for $j = 0, 1$ some powers of x in $V_j(x)$ have negative coefficients. This may be thought of as a bookkeeping device (explained in [31]) which allows for easily keeping track of fields which are eliminated by the requirement of imposing equations of motion or conservation laws. One consequence of this is that the full partition function for the module $\mathcal{V}_j$ (including descendants) may be calculated naively, with derivatives treated as if they acted freely, without worrying about equations of motion or conservation laws. The partition function for the module $\mathcal{V}_j$ is therefore simply

$$\mathrm{Tr}_{\mathcal{V}_j}[x^{D_0}] = \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[P_j x^{D_0}] = \frac{V_j(x)}{(1-x)^4}, \tag{5.8}$$

where we have defined P_j to be the projection operator $P_j : \mathcal{A} \times \mathcal{A} \to \mathcal{V}_j$.

The P_j form a complete set of orthogonal projection operators, so

$$\sum_{j=0}^{\infty} P_j = 1. \tag{5.9}$$

This implies the identity

$$\sum_{j=0}^{\infty} \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[P_j x^{D_0}] = \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[x^{D_0}] = z(x)^2, \tag{5.10}$$

which is indeed satisfied by (5.7) and (5.8).

5.2. Simple trace $\langle D_2(x)\rangle$

Here we calculate the expectation value (3.10) for the one-loop dilatation operator D_2. The calculation only takes one line since we have all of the machinery in place. In [30] it was shown that the eigenvalue of D_2 in the module $\mathcal{V}_j$ is simply the harmonic number

$$h(j) = \sum_{n=1}^{j} \frac{1}{n}, \qquad h(0) \equiv 0, \tag{5.11}$$

and therefore that D_2 may be written as

$$D_2 = \sum_{j=0}^{\infty} h(j) P_j. \tag{5.12}$$

From (5.8) and (5.12) we immediately have

$$\langle D_2(x) \rangle = \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[x^{D_0} D_2] = \sum_{j=0}^{\infty} h(j) \frac{V_j(x)}{(1-x)^4}. \tag{5.13}$$

Plugging in (5.7) and performing the sum over j gives

$$\langle D_2(x) \rangle = \frac{(1+\sqrt{x})^2}{(1-\sqrt{x})^6} \left[-(1-4\sqrt{x}+x)^2 \ln(1-x) - x(1-8\sqrt{x}+2x) \right]. \tag{5.14}$$

5.3. Permuted trace $\langle PD_2(w,y) \rangle$

As mentioned above, the calculation of $\langle PD_2(w,y) \rangle$ is complicated by the fact the two-letter $PSL(4|4)$ Casimir operator $J^2_{(12)}$ commutes only with the sum $D_{0(1)} + D_{0(2)}$, but not with $D_{0(1)}$ and $D_{0(2)}$ separately. In particular, it does not commute with $w^{D_{0(1)}} y^{D_{0(2)}}$, so the beautiful decomposition into the modules $\mathcal{V}_j$ that we employed in the previous subsection is of no use here. This apparent breaking of $PSL(4|4)$ is merely an artifact of our choice to simplify the calculation of $\mathrm{Tr}_L[\mathcal{P} x^{D_0} D_2]$ by tracing out $L-2$ sites on the chain and expressing (3.35) in terms of the two remaining sites.

A manifestly $PSL(4|4)$-invariant calculation of $\mathrm{Tr}_L[\mathcal{P} x^{D_0} D_2]$ would proceed as follows. First, one would need to know the collection $\mathcal{C}_L$ of $PSL(4|4)$ modules which appear in higher powers of the singleton representation $\mathcal{A}$,

$$\mathrm{Tr}(\mathcal{A} \times \cdots \times \mathcal{A}) = \mathrm{Tr}(\mathcal{A}^L) = \sum_{\mathcal{I} \in \mathcal{C}_L} \mathcal{V}_{\mathcal{I}} \tag{5.15}$$

(where Tr denotes the projection onto singlets of the cyclic group $\mathbb{Z}_L$). Then $PSL(4|4)$ invariance guarantees that in each resulting module $\mathcal{V}_{\mathcal{I}}$ the dilatation operator D_2 is proportional to the identity operator, with some calculable eigenvalue $h_{\mathcal{I}}$. The desired trace would then be given by

$$\mathrm{Tr}[x^{D_0} D_2] = \sum_{L=2}^{\infty} \sum_{\mathcal{I} \in \mathcal{C}_L} h_{\mathcal{I}} \mathrm{Tr}[P_{\mathcal{I}} x^{D_0}], \tag{5.16}$$

where $P_{\mathcal{I}}$ is the projection operator from $\mathrm{Tr}(\mathcal{A}^L)$ onto $\mathcal{V}_{\mathcal{I}}$.

The decomposition of $\mathcal{A}\times\mathcal{A}\times\mathcal{A}$ appears in [32] (see also [9]), where it was used to determine the one-loop anomalous dimensions of some operators consisting of three elementary fields. However, it seems quite challenging to implement the procedure outlined in the previous paragraph for arbitrary L, although of course it would be very interesting to do so.

Instead, we will proceed by using the matrix elements of the operator D_2, written down in [30] in a $GL(4|4)$ oscillator basis, and then calculating the desired trace 'by hand'. This is quite a lengthy calculation, so we begin by presenting the result. The interested reader can find more details below. We find:

$$\begin{aligned}\langle PD_2(w,y)\rangle = \frac{1+\sqrt{w}}{(1-\sqrt{w})^3}\frac{1+\sqrt{y}}{(1-\sqrt{y})^3}\frac{1}{(1+\sqrt{wy})^3(\sqrt{w}+\sqrt{y})^2}\Bigg[&wy(1-w)(1-y)p_1(w,y)\\ &+\frac{1}{2}\ln\left[\frac{(1-w)(1-y)}{(1-wy)^2}\right](1-\sqrt{wy})(\sqrt{w}+\sqrt{y})^2p_2(w,y)\\ &-\frac{1}{2}\frac{(\sqrt{w}-\sqrt{y})}{(\sqrt{w}+\sqrt{y})}\ln\left[\frac{1-w}{1-y}\right](1+\sqrt{wy})^3p_3(w,y)\Bigg],\end{aligned} \tag{5.17}$$

in terms of the three polynomials

$$\begin{aligned}p_1 = 4 &- 16w^{1/2} + 7w - 16y^{1/2} + 22w^{1/2}y^{1/2} - 16wy^{1/2}\\ &+ w^{3/2}y^{1/2} + 7y - 16w^{1/2}y + 6wy + w^{1/2}y^{3/2},\end{aligned} \tag{5.18}$$

$$\begin{aligned}p_2 = 1 &- 4w^{1/2} + w - 4y^{1/2} + 20w^{1/2}y^{1/2} - 20wy^{1/2} + 4w^{3/2}y^{1/2} + y - 20w^{1/2}y\\ &+ 42wy - 20w^{3/2}y + w^2y + 4w^{1/2}y^{3/2} - 20wy^{3/2} + 20w^{3/2}y^{3/2}\\ &- 4w^2y^{3/2} + wy^2 - 4w^{3/2}y^2 + w^2y^2,\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}p_3 = w &- 4w^{3/2} + w^2 + 4w^{1/2}y^{1/2} - 20wy^{1/2} + 20w^{3/2}y^{1/2} - 4w^2y^{1/2} + y\\ &- 20w^{1/2}y + 42wy - 20w^{3/2}y + w^2y - 4y^{3/2} + 20w^{1/2}y^{3/2}\\ &- 20wy^{3/2} + 4w^{3/2}y^{3/2} + y^2 - 4w^{1/2}y^2 + wy^2.\end{aligned} \tag{5.20}$$

Note the interesting identity

$$p_3(w,y) = y^2p_2(1/y,w) = w^2p_2(y,1/w). \tag{5.21}$$

It is useful to subject this complicated result to a simple check. When we set $w = y = x$, then the calculation of $\langle PD_2(x,x)\rangle$ can be done using the group theoretic techniques of the previous subsection. In particular, we have

$$\begin{aligned}\langle PD_2(x,x)\rangle &= \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[Px^{D_0}D_2] \\ &= \sum_{j=0}^{\infty} \mathrm{Tr}_{\mathcal{V}_j}[Px^{D_0}D_2] \\ &= \sum_{j=0}^{\infty}(-1)^j h(j)\frac{V_j(x)}{(1-x)^4},\end{aligned} \tag{5.22}$$

where we used the fact that the permutation operator P acts as $(-1)^j$ in $\mathcal{V}_j$ [33,34]. If we now substitute the expressions for $V_j(x)$ from (5.7) and perform the sum over j, we obtain

$$\begin{aligned}\langle PD_2(x,x)\rangle = \frac{(1+\sqrt{x})^3}{(1-\sqrt{x})^4}\frac{1}{(1+x)^3}\Big[&x(1-7x^{1/2}+x+x^{3/2}-6x^2+2x^{5/2}) \\ &-(1-7x^{1/2}+15x-25x^{3/2}+25x^2-15x^{3/2}+7x^3-x^{7/2})\ln(1+x)\Big].\end{aligned} \tag{5.23}$$

Encouragingly, this expression agrees precisely with that obtained by setting $w = y = x$ in (5.17).

Now we begin in earnest the calculation of (5.17). The first step is to use the matrix elements of D_2 in a $GL(4|4)$ oscillator basis, as presented in [30], to write down a sum which gives (5.17):

$$\begin{aligned}\langle PD_2(w,y)\rangle = \sum_{s_1,s_2,p_1,p_2,k=0}^{\infty}\sum_{F_1,F_2,j=0}^{4}&(-1)^j\binom{4}{j}\binom{4-j}{F_1-j}\binom{4-j}{F_2-j}\frac{p_1!p_2!}{k!(k+1)!} \\ &\times c(n_1+n_2, n_1-k-j, n_2-k-j)w^{1+s_1/2+p_1/2}y^{1+s_2/2+p_2/2} \\ &\times \mathcal{F}(-1-k,-k,-s_1,-s_2;2,1-k+p_1,1-k+p_2;1) \\ &\times \prod_{i=1}^{2}\delta(1-\tfrac{1}{2}s_i+\tfrac{1}{2}p_i-\tfrac{1}{2}F_i)(1+s_i)(1+p_i),\end{aligned} \tag{5.24}$$

where $\mathcal{F}$ is the regularized hypergeometric function,

$$n_i = s_i + p_i + F_i, \tag{5.25}$$

and the coefficients c are the matrix elements of D_2 given in [30]:

$$\begin{aligned}c(n,n_{12},n_{21}) &= \frac{1}{2}(-1)^{1+n_{12}n_{21}}\frac{\Gamma(\frac{1}{2}(n_{12}+n_{21}))\Gamma(1+\frac{1}{2}(n-n_{12}-n_{21}))}{\Gamma(1+\frac{1}{2}n)}, \\ c(n,0,0) &= \tfrac{1}{2}h(\tfrac{1}{2}n).\end{aligned} \tag{5.26}$$

The detailed derivation of (5.24), which is not entirely straightforward, is presented in appendix B. The next several steps of the calculation will be shown schematically. The skeptical reader is free to verify that the power series expansion of (5.24) agrees with that of (5.17) to any desired order.

After several manipulations similar to the ones in appendix B, equation (5.24) can be cast into the form

$$\begin{aligned}\langle PD_2(w,y)\rangle = \sum_{n=0}^{\infty}(-1)^n(n+1)(n+3)w^{1+n/2}y^{-n/2} \\ \times\left[\frac{(1+\sqrt{y})^2}{(1-\sqrt{y})^2}P_n(y)+(1+(-1)^{n+1}y^{1+n/2})^2\frac{\ln(1-y)}{y}\right],\end{aligned} \tag{5.27}$$

where $P_n(y)$ is a polynomial in $\sqrt{y}$ whose highest term is $\mathcal{O}(y^{n+1})$. We did not obtain an explicit formula for $P_n(y)$, though presumably it could be reverse engineered from the final answer (5.17). Instead, we use a trick by investigating the quantity

$$Q(w,y) = \langle PD_2(w,y)\rangle\left[\frac{(1-\sqrt{y})}{(1+\sqrt{y})}\frac{(1-\sqrt{w})}{(1+\sqrt{w})}\right]^2. \tag{5.28}$$

Now we use (5.27) and the power series expansion

$$\left(\frac{1-\sqrt{w}}{1+\sqrt{w}}\right)^2 = \sum_{m=1}^{\infty} d_m w^{m/2}, \qquad d_m = \delta_{m,0}+4m(-1)^m \tag{5.29}$$

to write

$$Q(w,y) = \sum_{m,n=0}^{\infty} d_m(-1)^n(n+1)(n+3)w^{1+n/2+m/2}\left(y^{-n/2}P_n(y)+W_n(y)\right), \tag{5.30}$$

where

$$W_n(y) = y^{-n/2}\left[\frac{1-\sqrt{y}}{1+\sqrt{y}}\right]^2(1+(-1)^{n+1}y^{1+n/2})^2\frac{\ln(1-y)}{y}. \tag{5.31}$$

The trick is now to break the power series expansion of $Q(w,y)$ into the upper diagonal terms (where the power of y is greater than the power of w), the diagonal terms, and the lower diagonal terms. Since $Q(w,y)$ is a symmetric function, the lower diagonal terms will be known once the upper diagonal terms are known. Furthermore, the diagonal terms can be extracted from (5.23), so all we have left to calculate are the upper diagonal terms. But since P_n is a polynomial whose highest-order term is y^{n+1}, we see from (5.30) that it

never contributes to the upper diagonal. Therefore, for purposes of computing the upper diagonal we are free to omit the $P_n(y)$ term in (5.30), and make the replacement

$$W_n(y) \to W_n(y)|_{y^k:k>1+n/2+m/2}\,, \tag{5.32}$$

where the notation means that we write out $W_n(y)$ as a series in y and throw away all powers of y less than or equal to $1+n/2+n/2$. This finally gives a formula for $Q(w,y)$ which is amenable to a calculation in Mathematica. In this manner, we obtain after a tedious but straightforward calculation the result (5.17).

6. One-Loop $\mathcal{N}=4$ SYM Partition Function

6.1. Single-trace

The complete one-loop correction to the partition function of $\mathcal{N}=4$ SYM theory on $\mathbb{R}\times S^3$ in the single-trace sector is given by substituting (5.14) and (5.17) into (3.35). We will not rewrite the formulas because no significant simplification seems to occur. Instead, let us note that the result has the expansion

$$\frac{\lambda \ln x}{4\pi^2}\mathrm{Tr}[x^{D_0}D_2] = \frac{\lambda \ln x}{4\pi^2}\left[3x^2+48x^{5/2}+384x^3+2064x^{7/2}+\cdots\right]. \tag{6.1}$$

(Each coefficient receives contributions from both kinds of traces, $\langle D_2\rangle$ and $\langle PD_2\rangle$.) The first term in (6.1) encodes the one-loop anomalous dimension of the Konishi operator, $3/4\pi^2$. The second term is 3×16, coming from 16 descendants of the Konishi operator. The third term is $384 = 6\times 2+20\times 3+104\times 3$, which come respectively from the 6 primary states $\mathrm{Tr}[\phi^I\phi^I\phi^J]$, which have anomalous dimension $1/2\pi^2$ according to table 3 in [30], the 20 Konishi descendants of the form $\mathrm{Tr}[\phi^I[\phi^J,\phi^K]]$, and finally 104 Konishi descendants which are traces of two elementary fields.

6.2. Multi-trace

Now let us extend the result of the previous subsection to the complete one-loop correction to the partition function, including multi-trace operators. As discussed in section 2.6, the diagonal matrix elements of the one-loop dilatation operator act additively on k-trace operators. Therefore, to go from the single-trace partition function to the multi-trace partition function we can still use the formula (2.13). Substituting

$$Z(x) = Z^{(0)}(x) + \frac{\lambda \ln x}{4\pi^2}Z^{(1)}(x) \tag{6.2}$$

and expanding to first order in λ, we find that the first order correction to the multi-trace partition function is

$$\mathcal{Z}^{(1)}(x) = \frac{\lambda \ln x}{4\pi^2} \mathcal{Z}^{(0)}(x) \sum_{n=1}^{\infty} Z^{(1)}(\omega^{n+1} x^n), \tag{6.3}$$

(a factor of $1/n$ is canceled by $\ln x^n = n \ln x$) where we recall that the tree level result $\mathcal{Z}^{(0)}(x)$ is written in (2.14).

Now let us plug the result from (3.35) into (6.3). The term proportional to $\langle PD_2 \rangle$ gives the sum

$$\sum_{n=1}^{\infty} \sum_{L=2}^{\infty} \sum_{k:(k,L)=1} \langle PD_2(\omega^{n(L-k)+1} x^{n(L-k)}, \omega^{nk+1} x^{nk}) \rangle = \sum_{a,b=1}^{\infty} \langle PD_2(\omega^{b+1} x^b, \omega^{a+1} x^a) \rangle. \tag{6.4}$$

To see why this equation is true, pick any positive a and b, and look at the left-hand side to see how many times (if any) the term $\langle PD_2(\omega^{a+1} x^a, \omega^{b+1} x^b) \rangle$ appears. This is equivalent to asking how many solutions, for given a and b, there are to the equations

$$nk = a, \qquad nL = a + b \tag{6.5}$$

for $L \geq 2$ and $k : (k, L) = 1$. The answer is that there is always precisely one solution: $n = (a, b)$, $L = (a + b)/n$ and $k = a/n$. Certainly if n were not the greatest common divisor of a and b but some smaller common divisor, then (6.5) would still give solutions for k and L, but these would not satisfy the constraint $(k, L) = 1$.

Next we plug the $\langle D_2(x) \rangle$ terms from (3.35) into (6.3), which gives the sum

$$\sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \phi(L) \frac{\langle D_2(\omega^{nL+1} x^{nL}) \rangle}{1 - z(\omega^{nL+1} x^{nL})} = \sum_{k=1}^{\infty} \left[\sum_{L|k} \phi(L) \right] \frac{\langle D_2(\omega^{k+1} x^k) \rangle}{1 - z(\omega^{k+1} x^k)} = \sum_{k=1}^{\infty} k \frac{\langle D_2(\omega^{k+1} x^k) \rangle}{1 - z(\omega^{k+1} x^k)}. \tag{6.6}$$

Combining (6.4) and (6.6) into (6.3) gives the final result

$$\mathcal{Z}^{(1)}(x) = \frac{\lambda \ln x}{4\pi^2} \mathcal{Z}^{(0)}(x) \left[\sum_{k=1}^{\infty} k \frac{\langle D_2(\omega^{k+1} x^k) \rangle}{1 - z(\omega^{k+1} x^k)} + \sum_{k,m=1}^{\infty} \langle PD_2(\omega^{k+1} x^k, \omega^{m+1} x^m) \rangle \right], \tag{6.7}$$

as advertised already in (1.1), for the one-loop partition function of $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$, expressed in terms of the free partition function $\mathcal{Z}^{(0)}$, the elementary partition function (2.4), and the traces (5.14) and (5.17).

6.3. One-loop Hagedorn temperature

The partition function $\mathcal{Z}(x)$ has a simple pole at the Hagedorn temperature

$$\mathcal{Z}(x) \sim \frac{c}{x_{\rm H} - x}, \tag{6.8}$$

where c is an irrelevant overall numerical coefficient. To compute the one-loop correction $\delta x_{\rm H}$ to the Hagedorn temperature, we simply expand

$$\frac{c}{x_{\rm H} + \delta x_{\rm H} - x} = \frac{c}{x_{\rm H} - x}\left[1 - \frac{\delta x_{\rm H}}{x_{\rm H} - x} + \cdots\right]. \tag{6.9}$$

When we compare this to (6.7) and recall that $x_{\rm H}$ is such that $z(x_{\rm H}) = 1$, we note that only the $k = 1$ term in the sum of $\langle D_2(x^k)\rangle$ contributes to the double pole at the Hagedorn temperature.[2] Reading off the residue of this pole, we find

$$\delta x_{\rm H} = -\lim_{x\to x_{\rm H}}\left[(x_{\rm H} - x)\frac{\lambda \ln x}{4\pi^2}\frac{\langle D_2(x)\rangle}{1 - z(x)}\right] = -\frac{2}{3}x_{\rm H}\frac{\lambda \ln x_{\rm H}}{4\pi^2}\langle D_2(x_{\rm H})\rangle. \tag{6.10}$$

Remarkably, we find from (5.14) that $\langle D_2(x_{\rm H})\rangle = 3/4$, which gives

$$\frac{\delta x_{\rm H}}{x_{\rm H}} = -\frac{\lambda \ln x_{\rm H}}{8\pi^2}, \qquad \text{and hence} \qquad \frac{\delta T_{\rm H}}{T_{\rm H}} = -\frac{1}{\ln x_{\rm H}}\frac{\delta x_{\rm H}}{x_{\rm H}} = \frac{\lambda}{8\pi^2}. \tag{6.11}$$

The one-loop Hagedorn temperature is therefore

$$T_{\rm H} = T_{\rm H}(0)\left(1 + \frac{\lambda}{8\pi^2} + \cdots\right), \qquad T_{\rm H}(0) = \frac{1}{\ln(7 + 4\sqrt{3})}. \tag{6.12}$$

It is encouraging that the sign is positive, consistent with the simple guess that the Hagedorn temperature is a monotonically increasing, smooth function of λ from $T_{\rm H} = T_0$ at zero coupling to the AdS/CFT prediction that $T_{\rm H} \sim \lambda^{1/4}$ at strong coupling.

[2] It is possible that the second term in (6.7) develops a pole at the Hagedorn temperature after evaluating the sum, in which case it would add a finite correction to the one-loop Hagedorn temperature. However, this does not happen in any of the subsectors that we studied, and numerical evidence suggests that it does not occur here either.

7. Discussion

In this paper we have presented, in (1.1), the one-loop correction to the partition function of $SU(N)$, $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$ at infinite N and below the Hagedorn temperature.

Several recent papers including [17,35,36,37] address the thermodynamics and phase transition structure of weakly coupled gauge theories with the goal of smoothly connecting onto the strong coupling predictions implied by the AdS/CFT correspondence. String theory in $AdS_5 \times S^5$ has both a Hagedorn transition [14] at $T_{\rm H} \sim \lambda^{1/4}$ and a Hawking-Page transition [38] involving the nucleation of AdS_5 black holes at $T_{\rm HP} = 3/2\pi$. The authors of [17] lay out the ways in which the phase diagram of the weakly coupled theory can be matched onto these strong coupling predictions. Various qualitatively different phase diagrams can in part be distinguished by the sign of a particular coefficient in the effective action for the Polyakov loop U, which is the order parameter for the phase transition. The computation of this coefficient requires a three-loop calculation in thermal Yang-Mills theory on S^3 which is in progress [39].

One of the motivations for the present work was the desire to provide an independent check of some pieces of the calculation of [39] from a completely orthogonal starting point. The one-loop calculation in this paper is equivalent to a two-loop calculation in thermal Yang-Mills theory and clearly has some, but not complete, overlap with the work of [39]. In one sense our calculation contains less information than the effective action for the Polyakov loop because we integrate out *all* degrees of freedom in the theory, including U. Moreover, our method is only applicable for temperatures below the Hagedorn transition. On the other hand, our calculation contains more information since we have also separately calculated, in (6.1), the one-loop correction to the partition function in the single-trace sector. It requires extra work because the 'trace' basis it not particularly natural at finite temperature, and the result is slightly messy because of the appearance of 'number theoretic' quantities, such as the Euler function $\phi(n)$ or the condition $k : (k, L) = 1$ in (3.35). These always disappear at the end of the day in any formula (such as (2.14) or (6.7)) which includes arbitrary-trace operators.

The finer information present in the single-trace result (6.1) is deeply related to recent studies of integrability in $\mathcal{N} = 4$ SYM theory because the one-loop correction to the partition function is essentially just the trace of the one-loop dilatation operator, and therefore encodes the sum of all anomalous dimensions in the theory (sorted according to

bare dimension by the powers of x^{D_0}). The one essential subtlety is that only cyclically invariant spin configurations correspond to gauge theory operators, because traces of elementary Yang-Mills fields are automatically cyclically invariant. Although the spin chain Hamiltonian only acts on two neighboring sites at a time, the projection onto cyclically invariant states induces an effective long-range interaction on the spin chain which is irrelevant at temperatures near the Hagedorn transition but dominates the partition function, and significantly complicates the calculation thereof, at low temperatures. Interestingly, the fact that the $PSL(4|4)$ Hamiltonian is integrable [6] played absolutely no apparent role in our calculation. It would be very interesting to understand our result in the context of integrability.

It would also be interesting to consider higher-loop corrections to our results. Although the dilatation operator of the full $\mathcal{N} = 4$ theory is only known to one loop, in the $SU(2)$ sector its precise form is known up to three loops (at the planar level). Furthermore, depending on the assumptions (such as integrability) that one is willing to make, one can go all the way to five loops [9].

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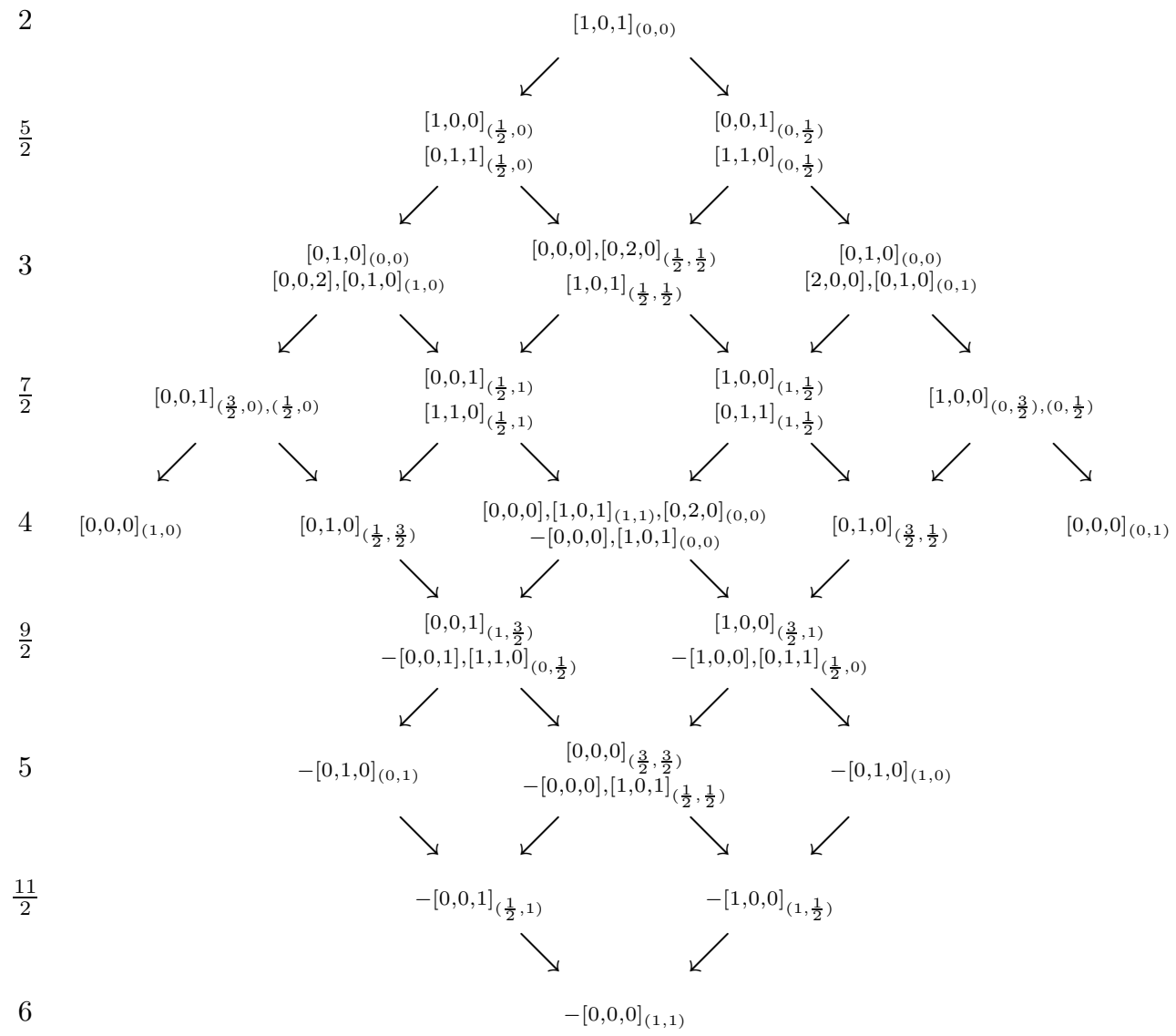
Appendix A. The Module $\mathcal{V}_1$

We tabulate here the $SL(4) \times SL(2) \times SL(2)$ decomposition of the primary states in the module $\mathcal{V}_1$, using the notation of [31] (where it is referred to as $\mathcal{B}^{\frac{1}{4},\frac{1}{4}}_{[1,0,1](0,0)}$).



With the help of the $SL(4) \times SL(2) \times SL(2)$ dimension formula

$$\dim[k,p,q]_{(j_1,j_2)} = \frac{1}{12}(k+p+q+3)(k+p+2)(p+q+2)(k+1)(p+1)(q+1)(2j_1+1)(2j_2+1) \tag{A.1}$$

we can immediately read off the partition function for primary states in $\mathcal{V}_1$,

$$V_1(x) = 15x^2 + 96x^{5/2} + 252x^3 + 336x^{7/2} + 210x^4 + 0x^{9/2} - 84x^5 - 48x^{11/2} - 9x^6, \tag{A.2}$$

in agreement with the result written in (5.7).

Appendix B. Some Details

In this appendix we show how to obtain the formula (5.24) from the matrix elements of D_2 given in [30].

B.1. The $GL(4|4)$ oscillator basis

The $GL(4|4)$ oscillator basis for $\mathcal{A}$ is realized by a set of four bosonic oscillators $\mathbf{a}^\alpha$, $\mathbf{b}^{\dot\alpha}$ ($\alpha, \dot\alpha \in \{1,2\}$) and four fermionic oscillators $\mathbf{c}^a$ ($a \in \{1,2,3,4\}$) with the usual relations

$$[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta] = \delta^\alpha_\beta, \qquad [\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}, \qquad \{\mathbf{c}^a, \mathbf{c}^\dagger_b\} = \delta^a_b, \tag{B.1}$$

and a vacuum $|0\rangle$ annihilated by all of the lowering operators. The only constraint on physical states is that they should be annihilated by the central charge

$$C = 1 - \frac{1}{2}\mathbf{a}^\dagger_\alpha \mathbf{a}^\alpha + \frac{1}{2}\mathbf{b}^\dagger_{\dot\alpha}\mathbf{b}^{\dot\alpha} - \frac{1}{2}\mathbf{c}^\dagger_a \mathbf{c}^a. \tag{B.2}$$

The tree-level dilatation operator corresponds to

$$D_0 = 1 + \frac{1}{2}\mathbf{a}^\dagger_\alpha \mathbf{a}^\alpha + \frac{1}{2}\mathbf{b}^\dagger_{\dot\alpha}\mathbf{b}^{\dot\alpha}. \tag{B.3}$$

To consider two letters $\mathcal{A}\times\mathcal{A}$ we simply have two copies of the above algebra, indexed by a subscript $(i) \in \{(1),(2)\}$. A general state in $\mathcal{A}_{(i)}$ will be labeled by its oscillator occupation numbers $(a^1_{(i)}, a^2_{(i)}, b^{\dot 1}_{(i)}, b^{\dot 2}_{(i)}, c^1_{(i)}, c^2_{(i)}, c^3_{(i)}, c^4_{(i)}) \equiv N_{(i)}$. In this basis we have

$$\begin{aligned} C_{(i)} &= 1 - \tfrac{1}{2}(a^1_{(i)} + a^2_{(i)}) + \tfrac{1}{2}(b^{\dot 1}_{(i)} + b^{\dot 2}_{(i)}) - \tfrac{1}{2}(c^1_{(i)} + c^2_{(i)} + c^3_{(i)} + c^4_{(i)}), \\ D_{0(i)} &= 1 + \tfrac{1}{2}(a^1_{(i)} + a^2_{(i)}) + \tfrac{1}{2}(b^{\dot 1}_{(i)} + b^{\dot 2}_{(i)}). \end{aligned} \tag{B.4}$$

Then to calculate a trace $\mathrm{Tr}_{\mathcal{A}_{(i)}}$ literally means that we perform a sum of the form

$$\sum_{\mathcal{A}_{(i)}} \equiv \sum_{N_{(i)}} \delta(C_{(i)}) = \sum_{a^1_{(i)}, a^2_{(i)}, b^{\dot 1}_{(i)}, b^{\dot 2}_{(i)} = 0}^{\infty} \; \sum_{c^1_{(i)}, c^2_{(i)}, c^3_{(i)}, c^4_{(i)} = 0}^{1} \delta(C_{(i)}) \tag{B.5}$$

over all possible oscillator numbers, subject to the physical state constraint. As a check, it is straightforward to confirm using this formula and (B.4) that

$$\mathrm{Tr}_{\mathcal{A}}[x^{D_0}] = \sum_{\mathcal{A}} x^{D_0} = \frac{2x(3-\sqrt{x})}{(1-\sqrt{x})^3}, \tag{B.6}$$

in agreement with (2.4).

Now we consider the action of the one-loop dilatation operator $D_{2(12)}$ on two sites, following [30]. If we let $\mathbf{A}_I^\dagger = (\mathbf{a}_\alpha^\dagger, \mathbf{b}_{\dot\alpha}^\dagger, \mathbf{c}_a^\dagger)$ schematically denote all of the raising operators, then a general state in $\mathcal{A}\times\mathcal{A}$ can be written as

$$|s_1,\ldots,s_n;\{I_i\}\rangle = \mathbf{A}^\dagger_{I_1(s_1)}\cdots\mathbf{A}^\dagger_{I_n(s_n)}|0\rangle, \tag{B.7}$$

where $s_i\in\{1,2\}$ indicates on which site the oscillator acts. The dilatation operator does not change the type or number of elementary $GL(4|4)$ oscillators, but can only cause them to hop from site 1 to 2 or vice versa according to the rule given in [30]:

$$D_{2(12)}|s_1,\ldots,s_n;\{I_i\}\rangle = \sum_{s_1',\ldots,s_n'=1,2} c(n,n_{12},n_{21})\delta(C_{(1)})\delta(C_{(2)})|s_1',\ldots,s_n';\{I_i\}\rangle, \tag{B.8}$$

where n_{12}, n_{21} count the number of oscillators hopping from site 1 to 2 or vice versa and the coefficients $c(n,n_{12},n_{21})$ are given in (5.26). In what follows we will consider a generalized operator of the form (B.8),

$$Q|s_1,\ldots,s_n;\{I_i\}\rangle = \sum_{s_1',\ldots,s_n'=1,2} q^n q_{12}^{n_{12}} q_{21}^{n_{21}}|s_1',\ldots,s_n';\{I_i\}\rangle, \tag{B.9}$$

with matrix elements $q^n q_{12}^{n_{12}} q_{21}^{n_{21}}$ instead of $c(n,n_{12},n_{21})$.

B.2. The combinatorics of hopping

We are therefore interested in the studying the combinatorics of oscillators hopping between two sites. Consider first a toy system with just a single bosonic oscillator $\mathbf{a}$. The most general state in $\mathcal{A}_{(1)}\times\mathcal{A}_{(2)}$ would then be

$$|a_{(1)},a_{(2)}\rangle = (\mathbf{a}^\dagger_{(1)})^{a_{(1)}}(\mathbf{a}^\dagger_{(2)})^{a_{(2)}}|0\rangle = \prod_{i=1}^n a^\dagger_{(s_i)}|0\rangle, \tag{B.10}$$

with

$$n = a_{(1)}+a_{(2)}, \qquad s_i = \{\underbrace{1,\ldots,1}_{a_{(1)}},\underbrace{2,\ldots,2}_{a_{(2)}}\}. \tag{B.11}$$

A quantity of interest is the matrix element

$$\begin{aligned} h_{a_{(1)},a_{(2)}}(q,q_{12},q_{21}) &= \langle a_{(1)},a_{(2)}|PQ|a_{(1)},a_{(2)}\rangle \\ &= \sum_{s_1',\ldots,s_n'=1}^{2} q^n q_{12}^{n_{12}} q_{21}^{n_{21}}\langle a_{(1)},a_{(2)}|P\prod_{i=1}^n a^\dagger_{(s_i')}|0\rangle, \end{aligned} \tag{B.12}$$

where Q is defined in (B.9). The function h counts the number of ways that the initial state $|a_{(1)}, a_{(2)}\rangle$ is mapped to itself, up to a permutation P, under the action of an operator of the form (B.8), weighted according to the powers of q_{12} and q_{21} which tell us the number of oscillators that have hopped from 1 to 2 or vice versa. An elementary combinatoric analysis reveals that

$$\begin{aligned} h_{a_{(1)},a_{(2)}}(q, q_{12}, q_{21}) &= \sum_{a=0}^{\infty} \binom{a_{(1)}}{a} \binom{a_{(2)}}{a} q^{a_{(1)}+a_{(2)}} q_{12}^{a_{(1)}-a} q_{21}^{a_{(2)}-a} \\ &= (qq_{12})^{a_{(1)}} (qq_{21})^{a_{(2)}} F(-a_{(1)}, -a_{(2)}, 1, (q_{12}q_{21})^{-1}), \end{aligned} \tag{B.13}$$

where F is the hypergeometric function.

A similar analysis can be done for the case of a single fermionic oscillator $\mathbf{c}$. For a general state $|c_{(1)}, c_{(2)}\rangle$ we find

$$g_{c_{(1)},c_{(2)}}(q, q_{(12)}, q_{(21)}) = (qq_{12})^{c_{(1)}} (qq_{21})^{c_{(2)}} \left(1 - \frac{c_{(1)}c_{(2)}}{q_{12}q_{21}}\right). \tag{B.14}$$

Since fermionic oscillators only have occupation numbers which are 0 or 1, this formula encodes the four cases

$$g_{0,0} = 1, \qquad g_{0,1} = qq_{21}, \qquad g_{1,0} = qq_{12}, \qquad g_{1,1} = q^2(q_{12}q_{21} - 1). \tag{B.15}$$

For $g_{0,0}$ there are no oscillators, so there is no hopping possible. For $g_{0,1}$ we start with one oscillator on site 2, which moves to site 1 giving a factor of q_{21}. In the final case, $g_{1,1}$ we have one fermionic oscillator on each site. They can either stay where they are, or they can flip, accounting for the two terms in $g_{1,1}$.

For a system with multiple oscillators we simply multiply together the appropriate partition functions (B.13) and (B.14) for the individual oscillators. For $GL(4|4)$ this gives

$$\begin{aligned} P_{N_{(1)},N_{(2)}}(q, q_{12}, q_{21}) &= \langle N_{(1)}, N_{(2)}|PQ|N_{(1)}, N_{(2)}\rangle \\ &= \prod_{\alpha=1}^{2} h(a_{(1)}^{\alpha}, a_{(2)}^{\alpha}) \prod_{\dot{\alpha}=1}^{2} h(b_{(1)}^{\dot{\alpha}}, b_{(2)}^{\dot{\alpha}}) \prod_{a=1}^{4} g(c_{(1)}^{a}, c_{(2)}^{a}). \end{aligned} \tag{B.16}$$

The quantity

$$\begin{aligned} R(w, y; q, q_{12}, q_{21}) &= \mathrm{Tr}_{\mathcal{A}\times\mathcal{A}}[Pw^{D_{0(1)}} y^{D_{0(2)}} Q] \\ &= \sum_{N_{(1)},N_{(2)}} \delta(C_{(1)})\delta(C_{(2)}) w^{D_{0(1)}} y^{D_{0(2)}} P_{N_{(1)},N_{(2)}}(q, q_{12}, q_{21}) \end{aligned} \tag{B.17}$$

then represents a trace over $\mathcal{A}\times\mathcal{A}$ which counts all of the possible hoppings between a state $|N_{(1)},N_{(2)}\rangle$ and its permutation $|N_{(2)},N_{(1)}\rangle$, weighted appropriately by w to the power of the dimension of site 1, times y to the power of the dimension of site 2, times q to the power of the total number of oscillators on both sites, times q_{12} to the power of the number of oscillators hopping from site 1 to 2, times q_{21} to the power of the number of oscillators hopping from site 2 to 1. Concretely, we obtain from (B.5), (B.13) and (B.14) the formula

$$\begin{aligned}R(w,y;q,q_{12},q_{21}) = &\sum_{a^1_{(1)},a^2_{(1)},b^{\dot 1}_{(1)},b^{\dot 2}_{(1)},a^1_{(2)},a^2_{(2)},b^{\dot 1}_{(2)},b^{\dot 2}_{(2)}=0}^{\infty}\ \sum_{c^1_{(1)},c^2_{(1)},c^3_{(1)},c^4_{(1)},c^1_{(2)},c^2_{(2)},c^3_{(2)},c^4_{(2)}=0}^{1}\\ &\delta(C_{(1)})\delta(C_{(2)})(qq_{12})^{n_{(1)}}(qq_{21})^{n_{(2)}}w^{D_{0(1)}}y^{D_{0(2)}}\prod_{a=1}^{4}\left(1-\frac{c^a_{(1)}c^a_{(2)}}{q_{12}q_{21}}\right)\\ &\times\prod_{\alpha=1}^{2}F(-a^\alpha_{(1)},-a^\alpha_{(2)},1,(q_{12}q_{21})^{-1})\prod_{\dot\alpha=1}^{2}F(-b^{\dot\alpha}_{(1)},-b^{\dot\alpha}_{(2)},1,(q_{12}q_{21})^{-1})\end{aligned} \tag{B.18}$$

where

$$n_{(i)}=\sum_{\alpha=1}^{2}a^\alpha_{(i)}+\sum_{\dot\alpha=1}^{2}b^{\dot\alpha}_{(i)}+\sum_{a=1}^{4}c^a_{(i)} \tag{B.19}$$

denotes the total number of oscillators on site i.

The first step in simplifying (B.18) is to rearrange the bosonic sums according to

$$\sum_{a^1_{(i)},a^2_{(i)}=0}^{\infty}f(a^1_{(i)},a^2_{(i)})=\sum_{s_{(i)}=0}^{\infty}\sum_{t_{(i)}=0}^{s_{(i)}}f(t_{(i)},s_{(i)}-t_{(i)}). \tag{B.20}$$

(For the b oscillators we will use $\dot s$ and $\dot t$ as the new summation variables.) After this substitution, the t variables only appear in the arguments of the hypergeometric functions. The sum over $t_{(1)}$ and $t_{(2)}$ can be done with the help of the identity

$$\begin{aligned}\sum_{t_{(1)}=0}^{s_{(1)}}\sum_{t_{(2)}=0}^{s_{(2)}}&F(-t_{(1)},-t_{(2)},1,z)F(-s_{(1)}+t_{(1)},-s_{(2)}+t_{(2)},1,z)\\ &=(1+s_{(1)})(1+s_{(2)})F(-s_{(1)},-s_{(2)},2,z).\end{aligned} \tag{B.21}$$

The sum over fermionic occupation numbers can be similarly simplified with the formula

$$\begin{aligned}\sum_{c^1_{(1)},c^2_{(1)},c^3_{(1)},c^4_{(1)},c^1_{(2)},c^2_{(2)},c^3_{(2)},c^4_{(2)}=0}^{1}&f\left(\sum_{a=1}^{4}c^a_{(1)},\sum_{a=1}^{4}c^a_{(2)}\right)\prod_{a=1}^{4}(1-c^a_{(1)}c^a_{(2)}z)\\ &=\sum_{F_{(1)}=0}^{4}\sum_{F_{(2)}=0}^{4}f(F_{(1)},F_{(2)})\sum_{j=0}^{4}(-1)^jz^j\binom{4}{j}\binom{4-j}{F_{(1)}-j}\binom{4-j}{F_{(2)}-j}.\end{aligned} \tag{B.22}$$

The results (B.20) and (B.22) allow (B.18) to be written as

$$\begin{aligned} R(w,y;q,q_{12},q_{21}) = & \sum_{s_{(1)},s_{(2)},\dot{s}_{(1)},\dot{s}_{(2)}=0}^{\infty} \sum_{F_{(1)},F_{(2)},j=0}^{4} (-1)^j z^j \binom{4}{j}\binom{4-j}{F_{(1)}-j}\binom{4-j}{F_{(2)}-j} \\ & \times w^{1+s_{(1)}/2+\dot{s}_{(1)}/2} y^{1+s_{(2)}/2+\dot{s}_{(2)}/2} \\ & \times (qq_{12})^{n_{(1)}} (qq_{21})^{n_{(2)}} F(-s_{(1)},-s_{(2)},2,z) F(-\dot{s}_{(1)},-\dot{s}_{(2)},2,z) \\ & \times \prod_{i=1}^{2} \delta(1-\tfrac{1}{2}s_{(i)}+\tfrac{1}{2}\dot{s}_{(i)}-\tfrac{1}{2}F_{(i)})(1+s_{(i)})(1+\dot{s}_{(i)}), \end{aligned} \tag{B.23}$$

where

$$z = \frac{1}{q_{12}q_{21}}, \qquad n_{(i)} = s_{(i)} + \dot{s}_{(i)} + F_{(i)}. \tag{B.24}$$

Of course, we are not particularly interested in the quantity $R(w,y;q,q_{12},q_{21})$. It is simply an auxiliary quantity which counts the possible ways for the operators to hop between sites. We are interested in the trace of (B.8) instead of the trace of (B.9), which we can calculate via the replacement

$$\langle PD_2(w,y)\rangle = R(w,y;q,q_{12},q_{21})|_{q^n q_{12}^{n_{12}} q_{21}^{n_{21}} \to c(n,n_{12},n_{21})}\,, \tag{B.25}$$

where the notation means simply that we expand (B.23) in powers of q, q_{12} and q_{21} and then make the substitution indicated for the various powers.

In order to proceed, we expose the powers of q_{12} and q_{21} in (B.23) by using the identity

$$\begin{aligned} & F(-s_{(1)},-s_{(2)},2,z)F(-\dot{s}_{(1)},-\dot{s}_{(2)},2,z) \\ & = \sum_{k=0}^{\infty} z^k \frac{\dot{s}_{(1)}!\dot{s}_{(2)}!}{k!(k+1)!} \mathcal{F}(-1-k,-k,-s_{(1)},-s_{(2)};2,1-k+\dot{s}_{(1)},1-k+\dot{s}_{(2)};1) \end{aligned} \tag{B.26}$$

where $\mathcal{F}$ is the regularized hypergeometric function. Combining (B.25), (B.26) and (B.23) finally leads to the formula (5.24) for the desired trace, after some simplification of the notation.

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